Eur. Phys. J. C 50, 781–791 (2007)
DOI 10.1140/epjc/s10052-007-0249-z

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Analysis of the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ with light-cone QCD sum rules

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Received: 17 September 2006 / Revised version: 28 January 2007 /
Published online: 15 March 2007 –

Abstract. In this article, we calculate the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ within the framework of the light-cone QCD sum rule approach. The numerical values of $f^+_{K\pi}(Q^2)$ are compatible with existing theoretical calculations, and the central value of $f^+_{K\pi}(0)$ ($f^+_{K\pi}(0) = 0.97$) is in excellent agreement with the values from chiral perturbation theory and lattice QCD. The values of $|f^-_{K\pi}(0)|$ are very large compared to the theoretical calculations and experimental data, and they cannot give any reliable prediction. At large momentum transfer with $Q^2 > 5\,\mathrm{GeV}^2$, the form factors $f^+_{K\pi}(Q^2)$ and $|f^-_{K\pi}(Q^2)|$ can either show the asymptotic behavior of $\frac{1}{Q^2}$ or decrease more quickly than $\frac{1}{Q^2}$; more experimental data are needed to select the ideal sum rules.

PACS. 12.38.Lg; 12.38.Bx; 12.15.Hh

1 Introduction

Semileptonic $K \to \pi\ell\nu$ ($K_{\ell 3}$) decays provide the most precise determination of the Cabibbo–Kobayashi–Maskawa (CKM) matrix element $|V_{us}|$ [1]. The experimental input parameters are the semileptonic decay widths and the vector form factors $f^+_{K\pi}(q^2)$ and $f^-_{K\pi}(q^2)$, which are necessary in calculating the phase space integrals. The main uncertainty in the quantity $|V_{us} f^+_{K\pi}(0)|$ comes from the unknown shape of the hadronic form factor $f^+_{K\pi}(q^2)$, which is measurable at $m_l^2 < q^2 < (m_K - m_\pi)^2$ in $K_{\ell 3}$ decays or at $(m_K + m_\pi)^2 < q^2 < m_\tau^2$ in $\tau \to K\pi\nu$ decays. The experimental data can be fitted to the functions with either pole models or series expansions; however, systematic errors are introduced due to the different parameterizations. Conservation of the vector current implies $f^+_{K\pi}(0) = 1$ at zero momentum transfer [2], and another powerful theoretical constraint on $f^+_{K\pi}(0)$ is provided by the SU(3) symmetry of the light pseudoscalar mesons and the Ademollo–Gatto theorem [3, 4]. SU(3) symmetry breaking effects, $f^+_{K\pi}(0) - 1$, start at second order in $m_s - m_q$. Chiral perturbation theory (ChPT) provides a natural and powerful tool to take into account the SU(3) symmetry breaking effects due to the masses of the light quarks; $f^+_{K\pi}(q^2)$ is usually calculated by ChPT [5–9]. Presently, comparison of theory and experiment, and of different experiments among themselves, is complicated by the uncertainties in the form factor $f^+_{K\pi}(q^2)$ with zero momentum transfer.

In this article, we calculate the values of the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ within the framework of the light-cone QCD sum rule approach. The light-cone QCD sum rule approach carries out the operator product expansion near the light-cone, $x^2 \approx 0$, instead of the short distance $x \approx 0$, while the non-perturbative matrix elements are parameterized by the light-cone distribution amplitudes, which are classified according to their twists instead of the vacuum condensates [10–16]. The non-perturbative parameters in the light-cone distribution amplitudes are calculated by the conventional QCD sum rules and the values are universal [17–20].

The article is arranged as follows: in Sect. 2 we derive the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ with the light-cone QCD sum rule approach; in Sect. 3 numerical results are given and discussed, and in Sect. 4 one finds our conclusions.

2 Vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ with light-cone QCD sum rules

In the following, we write down the definitions for the vector form factors $f^+_{K\pi}(q^2)$ and $f^-_{K\pi}(q^2)$:

$$\begin{aligned}&\langle \pi(q+p)|J_\mu(0)|K(p)\rangle \\ &\quad = 2f^+_{K\pi}(q^2)p_\mu + \left\{f^+_{K\pi}(q^2) - f^-_{K\pi}(q^2)\right\}q_\mu\,, \qquad (1)\end{aligned}$$

where $J_\mu(x)$ is the vector current. We study the vector form factors $f^+_{K\pi}(q^2)$ and $f^-_{K\pi}(q^2)$ with the two-point cor-

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relation functions $\Pi^A_\mu(p,q)$ and $\Pi^B_\mu(p,q)$,

$$\Pi^A_\mu(p,q)=\mathrm{i}\int \mathrm{d}^4x\mathrm{e}^{-\mathrm{i}q\cdot x}\langle 0|T\{J_K(0)J^A_\mu(x)\}|\pi(p)\rangle\,, \quad (2)$$

$$\Pi^B_\mu(p,q)=\mathrm{i}\int \mathrm{d}^4x\mathrm{e}^{-\mathrm{i}q\cdot x}\langle 0|T\{J_\pi(0)J^B_\mu(x)\}|K(p)\rangle\,, \quad (3)$$

$$\begin{aligned}
J^A_\mu(x)&=\bar{s}(x)\gamma_\mu u(x)\,,\\
J^B_\mu(x)&=\bar{d}(x)\gamma_\mu s(x)\,,\\
J_K(x)&=\bar{d}(x)\mathrm{i}\gamma_5 s(x)\,,\\
J_\pi(x)&=\bar{u}(x)\mathrm{i}\gamma_5 d(x)\,,
\end{aligned} \quad (4)$$

where $J_K(x)$ and $J_\pi(x)$ interpolate the K and π mesons, respectively, and we choose the pseudoscalar currents so as to avoid the possible contaminations from the axial-vector mesons. The correlation functions $\Pi^{A(B)}_\mu(p,q)$ can be decomposed as follows:

$$\begin{aligned}
\Pi^{A(B)}_\mu(p,q)&=\Pi^{A(B)}_p(q^2,(q+p)^2)p_\mu\\
&\quad+\Pi^{A(B)}_q(q^2,(q+p)^2)q_\mu\,,
\end{aligned} \quad (5)$$

due to Lorentz covariance. In this article, we derive the sum rules with the tensor structures p_μ and q_μ, respectively.

According to the basic assumption of current–hadron duality in the QCD sum rule approach [17–20], we can insert a complete series of intermediate states with the same quantum numbers as the current operators $J_K(x)$ and $J_\pi(x)$ into the correlation functions Π^A_μ and Π^B_μ to obtain the hadronic representation. After isolating the ground state contributions from the pole terms of the K and π mesons, the correlation functions Π^A_μ and Π^B_μ can be expressed in the following forms:

$$\begin{aligned}
\Pi^A_\mu(p,q)&=\frac{2f_K m_K^2 f^+_{K\pi}(q^2)}{(m_d+m_s)\{m_K^2-(q+p)^2\}}p_\mu\\
&\quad+\frac{f_K m_K^2\{f^+_{K\pi}(q^2)-f^-_{K\pi}(q^2)\}}{(m_d+m_s)\{m_K^2-(q+p)^2\}}q_\mu+\dots\,, \quad (6)\\
\Pi^B_\mu(p,q)&=\frac{2f_\pi m_\pi^2 f^+_{K\pi}(q^2)}{(m_d+m_u)\{m_\pi^2-(q+p)^2\}}p_\mu\\
&\quad+\frac{f_\pi m_\pi^2\{f^+_{K\pi}(q^2)-f^-_{K\pi}(q^2)\}}{(m_d+m_u)\{m_\pi^2-(q+p)^2\}}q_\mu+\dots\,; \quad (7)
\end{aligned}$$

here we have not shown the contributions from the high resonances and continuum states explicitly. They are suppressed after Borel transformation and subtraction. We use the standard definitions for the weak decay constants f_K and f_π,

$$\begin{aligned}
\langle 0|J_K(0)|K(q)\rangle&=\frac{f_K m_K^2}{m_s+m_d}\,,\\
\langle 0|J_\pi(0)|\pi(q)\rangle&=\frac{f_\pi m_\pi^2}{m_u+m_d}\,.
\end{aligned}$$

In the following, we briefly outline the operator product expansion for the correlation functions Π^A_μ and Π^B_μ in perturbative QCD theory. The calculations are performed at the large space-like momentum regions $P^2=-(q+p)^2\gg 0$ and $Q^2=-q^2\gg 0$, which correspond to the small light-cone distance $x^2\approx 0$ required by the validity of the operator product expansion approach. We write down the propagator of a massive quark in the external gluon field in the Fock–Schwinger gauge first [21]:

$$\begin{aligned}
&\langle 0|T\{q_i(x_1)\,\bar{q}_j(x_2)\}|0\rangle\\
&\quad=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k(x_1-x_2)}\\
&\qquad\times\left\{\frac{\not k+m}{k^2-m^2}\delta_{ij}-\int_0^1\mathrm{d}v g_s G^{\mu\nu}_{ij}(vx_1+(1-v)x_2)\right.\\
&\qquad\times\left.\left[\frac{1}{2}\frac{\not k+m}{(k^2-m^2)^2}\sigma_{\mu\nu}-\frac{1}{k^2-m^2}v(x_1-x_2)_\mu\gamma_\nu\right]\right\}, \quad (8)
\end{aligned}$$

where $G_{\mu\nu}$ is the gluonic field strength, and g_s denotes the strong coupling constant. Substituting the above s, d quark propagators and the corresponding π, K mesons light-cone distribution amplitudes into the correlation functions Π^A_μ and Π^B_μ in (2) and (3), and completing the integrals over the variables x and k, finally we obtain the representations at the level of quark–gluon degrees of freedom,

$$\begin{aligned}
\Pi^A_p&=\frac{f_\pi m_\pi^2}{m_u+m_d}\int_0^1\mathrm{d}u\frac{u\phi_p(u)}{m_s^2-(q+up)^2}\\
&\quad-m_s f_\pi m_\pi^2\int_0^1\mathrm{d}u\int_0^u\mathrm{d}t\frac{uB(t)}{\{m_s^2-(q+up)^2\}^2}\\
&\quad+\frac{1}{6}\frac{f_\pi m_\pi^2}{m_u+m_d}\int_0^1\mathrm{d}u\phi_\sigma(u)\\
&\quad\times\left\{\left[1-u\frac{\mathrm{d}}{\mathrm{d}u}\right]\frac{1}{m_s^2-(q+up)^2}+\frac{2m_s^2}{\left[m_s^2-(q+up)^2\right]^2}\right\}\\
&\quad+m_s f_\pi\int_0^1\mathrm{d}u\\
&\quad\times\left\{\frac{\phi_\pi(u)}{m_s^2-(q+up)^2}-\frac{m_\pi^2m_s^2}{2}\frac{A(u)}{\left[m_s^2-(q+up)^2\right]^3}\right\}\\
&\quad-f_{3\pi}\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u T(\alpha_d,\alpha_g,\alpha_u)\\
&\quad\times\left\{\frac{(1+2v)um_\pi^2}{\left[m_s^2-(q+up)^2\right]^2}\right.\\
&\qquad\left.\left.-2(1-v)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{m_s^2-(q+up)^2}\right\}\right|_{u=(1-v)\alpha_g+\alpha_u}\\
&\quad+4m_s f_\pi m_\pi^4\int_0^1\mathrm{d}vv\int_0^1\mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha\\
&\quad\times\left.\frac{u\Phi(1-\alpha-\beta,\beta,\alpha)}{\{m_s^2-(q+up)^2\}^3}\right|_{1-v\alpha_g}\\
&\quad-4m_s f_\pi m_\pi^4\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u\int_0^{\alpha_u}\mathrm{d}\alpha\\
&\quad\times\left.\frac{u\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)}{\{m_s^2-(q+up)^2\}^3}\right|_{u=(1-v)\alpha_g+\alpha_u}
\end{aligned}$$

$$
\begin{aligned}
&+m_s f_\pi m_\pi^2\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u\\
&\times\left.\frac{\Psi(\alpha_d,\alpha_g,\alpha_u)}{\left\{m_s^2-(q+up)^2\right\}^2}\right|_{u=(1-v)\alpha_g+\alpha_u}, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\Pi_p^B=&\frac{f_K m_K^2}{m_u+m_s}\int_0^1 \mathrm{d}u\frac{u\phi_p(u)}{m_d^2-(q+up)^2}\\
&-m_d f_K m_K^2\int_0^1\mathrm{d}u\int_0^u \mathrm{d}t\frac{uB(t)}{\left\{m_d^2-(q+up)^2\right\}^2}\\
&+\frac{1}{6}\frac{f_K m_K^2}{m_u+m_s}\int_0^1\mathrm{d}u\phi_\sigma(u)\\
&\times\left\{\left[1-u\frac{\mathrm{d}}{\mathrm{d}u}\right]\frac{1}{m_d^2-(q+up)^2}+\frac{2m_d^2}{\left[m_d^2-(q+up)^2\right]^2}\right\}\\
&+m_d f_K\int_0^1\mathrm{d}u\\
&\times\left\{\frac{\phi_K(u)}{m_d^2-(q+up)^2}-\frac{m_K^2m_d^2}{2}\frac{A(u)}{\left[m_d^2-(q+up)^2\right]^3}\right\}\\
&-f_{3K}\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s T(\alpha_u,\alpha_g,\alpha_s)\\
&\times\left\{\frac{(1+2v)um_K^2}{\left[m_d^2-(q+up)^2\right]^2}\right.\\
&\qquad\left.\left.-2(1-v)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{m_d^2-(q+up)^2}\right\}\right|_{u=(1-v)\alpha_g+\alpha_s}\\
&+4m_d f_K m_K^4\int_0^1\mathrm{d}vv\int_0^1\mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha\\
&\times\left.\frac{u\Phi(1-\alpha-\beta,\beta,\alpha)}{\left\{m_d^2-(q+up)^2\right\}^3}\right|_{1-v\alpha_g}\\
&-4m_d f_K m_K^4\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s\int_0^{\alpha_s}\mathrm{d}\alpha\\
&\times\left.\frac{u\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)}{\left\{m_d^2-(q+up)^2\right\}^3}\right|_{u=(1-v)\alpha_g+\alpha_s}\\
&+m_d f_K m_K^2\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s\\
&\times\left.\frac{\Psi(\alpha_u,\alpha_g,\alpha_s)}{\left\{m_d^2-(q+up)^2\right\}^2}\right|_{u=(1-v)\alpha_g+\alpha_s}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\Pi_q^A=&\frac{f_\pi m_\pi^2}{m_u+m_d}\int_0^1\mathrm{d}u\frac{\phi_p(u)}{m_s^2-(q+up)^2}\\
&-m_s f_\pi m_\pi^2\int_0^1\mathrm{d}u\int_0^u\mathrm{d}t\frac{B(t)}{\left\{m_s^2-(q+up)^2\right\}^2}\\
&-\frac{1}{6}\frac{f_\pi m_\pi^2}{m_u+m_d}\int_0^1\mathrm{d}u\phi_\sigma(u)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{m_s^2-(q+up)^2}\\
&-f_{3\pi}m_\pi^2\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u T(\alpha_d,\alpha_g,\alpha_u)\\
&\times\left.\frac{1+2v}{\left\{m_s^2-(q+up)^2\right\}^2}\right|_{u=(1-v)\alpha_g+\alpha_u}\\
&+4m_s f_\pi m_\pi^4\int_0^1\mathrm{d}vv\int_0^1\mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha\\
&\times\left.\frac{\Phi(1-\alpha-\beta,\beta,\alpha)}{\left\{m_s^2-(q+up)^2\right\}^3}\right|_{1-v\alpha_g}\\
&-4m_s f_\pi m_\pi^4\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u\int_0^{\alpha_u}\mathrm{d}\alpha\\
&\times\left.\frac{\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)}{\left\{m_s^2-(q+up)^2\right\}^3}\right|_{u=(1-v)\alpha_g+\alpha_u}, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\Pi_q^B=&\frac{f_K m_K^2}{m_u+m_s}\int_0^1\mathrm{d}u\frac{\phi_p(u)}{m_d^2-(q+up)^2}\\
&-m_d f_K m_K^2\int_0^1\mathrm{d}u\int_0^u\mathrm{d}t\frac{B(t)}{\left\{m_d^2-(q+up)^2\right\}^2}\\
&-\frac{1}{6}\frac{f_K m_K^2}{m_u+m_s}\int_0^1\mathrm{d}u\phi_\sigma(u)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{m_d^2-(q+up)^2}\\
&-f_{3K}m_K^2\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s T(\alpha_u,\alpha_g,\alpha_s)\\
&\times\left.\frac{1+2v}{\left\{m_d^2-(q+up)^2\right\}^2}\right|_{u=(1-v)\alpha_g+\alpha_s}\\
&+4m_d f_K m_K^4\int_0^1\mathrm{d}vv\int_0^1\mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha\\
&\times\left.\frac{\Phi(1-\alpha-\beta,\beta,\alpha)}{\left\{m_d^2-(q+up)^2\right\}^3}\right|_{1-v\alpha_g}\\
&-4m_d f_K m_K^4\int_0^1\mathrm{d}v\int_0^1\mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s\int_0^{\alpha_s}\mathrm{d}\alpha\\
&\times\left.\frac{\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)}{\left\{m_d^2-(q+up)^2\right\}^3}\right|_{u=(1-v)\alpha_g+\alpha_s}, \qquad (12)
\end{aligned}
$$

where $\Phi=A_\parallel+A_\perp-V_\perp-V_\parallel$ and $\Psi=2A_\perp-2V_\perp-A_\parallel+V_\parallel$. In the calculation, we have used the two-particle and three-particle K and π mesons light-cone distribution amplitudes [10–16, 21–23], and the explicit expressions of the K meson light-cone distribution amplitudes are presented in the appendix; the corresponding ones for the π meson can be obtained by simple substitution of the nonperturbative parameters. The parameters in the light-cone distribution amplitudes are scale dependent and can be estimated with the QCD sum rule approach [10–16, 21–23]. In this article, the energy scale μ is chosen to be $\mu=1$ GeV.

We take the Borel transformation with respect to the variable $P^2=-(q+p)^2$ for the correlation functions $\Pi_p^{A(B)}$ and $\Pi_q^{A(B)}$ and obtain the analytical expressions for those invariant functions. After matching with the hadronic representations below the thresholds, we obtain the following four sum rules for the form factors $f^+_{K\pi}(q^2)$ and $f^-_{K\pi}(q^2)$:

$$
\begin{aligned}
&\frac{2f_K m_K^2}{m_d+m_s}f^+_{K\pi}(q^2)\mathrm{e}^{-\frac{m_K^2}{M^2}}\\
&=\frac{f_\pi m_\pi^2}{m_u+m_d}\int_{\Delta_A}^1\mathrm{d}u\phi_p(u)\mathrm{e}^{-\mathrm{DD}}
\end{aligned}
$$

$$
\begin{aligned}
&-m_s f_\pi m_\pi^2 \int_{\Delta_A}^1 \mathrm{d}u \int_0^u \mathrm{d}t \frac{B(t)}{uM^2} \mathrm{e}^{-\mathrm{DD}} \\
&+\frac{1}{6}\frac{f_\pi m_\pi^2}{m_u+m_d}\int_{\Delta_A}^1 \mathrm{d}u\phi_\sigma(u)\left\{\left[1-u\frac{\mathrm{d}}{\mathrm{d}u}\right]\frac{1}{u}+\frac{2m_s^2}{u^2M^2}\right\}\mathrm{e}^{-\mathrm{DD}} \\
&+m_s f_\pi \int_{\Delta_A}^1 \mathrm{d}u\left\{\frac{\phi_\pi(u)}{u}-\frac{m_\pi^2 m_s^2 A(u)}{4u^3M^4}\right\}\mathrm{e}^{-\mathrm{DD}} \\
&-f_{3\pi}\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u T(\alpha_d,\alpha_g,\alpha_u)\Theta(u-\Delta_A) \\
&\times\left\{\frac{(1+2v)m_\pi^2}{uM^2}-2(1-v)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{u}\right\}\mathrm{e}^{-\mathrm{DD}}\Big|_{u=(1-v)\alpha_g+\alpha_u} \\
&+2m_s f_\pi m_\pi^4\int_0^1 \mathrm{d}v v\int_0^1 \mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\beta,\beta,\alpha)\Theta(u-\Delta_A)}{u^2M^4}\mathrm{e}^{-\mathrm{DD}}\Big|_{u=1-v\alpha_g} \\
&-2m_s f_\pi m_\pi^4\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u\int_0^{\alpha_u}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)\Theta(u-\Delta_A)}{u^2M^4}\mathrm{e}^{-\mathrm{DD}}\Big|_{u=(1-v)\alpha_g+\alpha_u} \\
&+m_s f_\pi m_\pi^2\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u \\
&\times\frac{\Psi(\alpha_d,\alpha_g,\alpha_u)\Theta(u-\Delta_A)}{u^2M^2}\mathrm{e}^{-\mathrm{DD}}\Big|_{u=(1-v)\alpha_g+\alpha_u}, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2f_\pi m_\pi^2}{m_u+m_d}f^+_{K\pi}(q^2)\mathrm{e}^{-\frac{m_\pi^2}{M^2}} \\
&=\frac{f_K m_K^2}{m_u+m_s}\int_{\Delta_B}^1 \mathrm{d}u\phi_p(u)\mathrm{e}^{-\mathrm{EE}} \\
&-m_d f_K m_K^2\int_{\Delta_B}^1 \mathrm{d}u\int_0^u \mathrm{d}t\frac{B(t)}{uM^2}\mathrm{e}^{-\mathrm{EE}} \\
&+\frac{1}{6}\frac{f_K m_K^2}{m_u+m_s}\int_{\Delta_B}^1 \mathrm{d}u\phi_\sigma(u)\left\{\left[1-u\frac{\mathrm{d}}{\mathrm{d}u}\right]\frac{1}{u}+\frac{2m_d^2}{u^2M^2}\right\}\mathrm{e}^{-\mathrm{EE}} \\
&+m_d f_K\int_{\Delta_B}^1 \mathrm{d}u\left\{\frac{\phi_K(u)}{u}-\frac{m_K^2 m_d^2 A(u)}{4u^3M^4}\right\}\mathrm{e}^{-\mathrm{EE}} \\
&-f_{3K}\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s T(\alpha_u,\alpha_g,\alpha_s)\Theta(u-\Delta_B) \\
&\times\left\{\frac{(1+2v)m_K^2}{uM^2}-2(1-v)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{u}\right\}\mathrm{e}^{-\mathrm{EE}}\Big|_{u=(1-v)\alpha_g+\alpha_s} \\
&+2m_d f_K m_K^4\int_0^1 \mathrm{d}v v\int_0^1 \mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\beta,\beta,\alpha)\Theta(u-\Delta_B)}{u^2M^4}\mathrm{e}^{-\mathrm{EE}}\Big|_{u=1-v\alpha_g} \\
&-2m_d f_K m_K^4\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s\int_0^{\alpha_s}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)\Theta(u-\Delta_B)}{u^2M^4}\mathrm{e}^{-\mathrm{EE}}\Big|_{u=(1-v)\alpha_g+\alpha_s} \\
&+m_d f_K m_K^2\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s \\
&\times\frac{\Psi(\alpha_u,\alpha_g,\alpha_s)\Theta(u-\Delta_B)}{u^2M^2}\mathrm{e}^{-\mathrm{EE}}\Big|_{u=(1-v)\alpha_g+\alpha_s}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{f_K m_K^2}{m_d+m_s}\{f^+_{K\pi}(q^2)-f^-_{K\pi}(q^2)\}\mathrm{e}^{-\frac{m_K^2}{M^2}} \\
&=\frac{f_\pi m_\pi^2}{m_u+m_d}\int_{\Delta_A}^1 \mathrm{d}u\frac{\phi_p(u)}{u}\mathrm{e}^{-\mathrm{DD}} \\
&-m_s f_\pi m_\pi^2\int_{\Delta_A}^1 \mathrm{d}u\int_0^u \mathrm{d}t\frac{B(t)}{u^2M^2}\mathrm{e}^{-\mathrm{DD}} \\
&-\frac{1}{6}\frac{f_\pi m_\pi^2}{m_u+m_d}\int_{\Delta_A}^1 \mathrm{d}u\phi_\sigma(u)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{u}\mathrm{e}^{-\mathrm{DD}} \\
&-f_{3\pi}m_\pi^2\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u T(\alpha_d,\alpha_g,\alpha_u) \\
&\times\Theta(u-\Delta_A)\frac{1+2v}{u^2M^2}\mathrm{e}^{-\mathrm{DD}}\Big|_{u=(1-v)\alpha_g+\alpha_u} \\
&+2m_s f_\pi m_\pi^4\int_0^1 \mathrm{d}v v\int_0^1 \mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\beta,\beta,\alpha)\Theta(u-\Delta_A)}{u^3M^4}\mathrm{e}^{-\mathrm{DD}}\Big|_{1-v\alpha_g} \\
&-2m_s f_\pi m_\pi^4\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_u\int_0^{\alpha_u}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)\Theta(u-\Delta_A)}{u^3M^4}\mathrm{e}^{-\mathrm{DD}}\Big|_{u=(1-v)\alpha_g+\alpha_u}, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{f_\pi m_\pi^2}{m_u+m_d}\{f^+_{K\pi}(q^2)-f^-_{K\pi}(q^2)\}\mathrm{e}^{-\frac{m_\pi^2}{M^2}} \\
&=\frac{f_K m_K^2}{m_u+m_s}\int_{\Delta_B}^1 \mathrm{d}u\frac{\phi_p(u)}{u}\mathrm{e}^{-\mathrm{EE}} \\
&-m_d f_K m_K^2\int_{\Delta_B}^1 \mathrm{d}u\int_0^u \mathrm{d}t\frac{B(t)}{u^2M^2}\mathrm{e}^{-\mathrm{EE}} \\
&-\frac{1}{6}\frac{f_K m_K^2}{m_u+m_s}\int_{\Delta_B}^1 \mathrm{d}u\phi_\sigma(u)\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{u}\mathrm{e}^{-\mathrm{EE}} \\
&-f_{3K}m_K^2\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s T(\alpha_u,\alpha_g,\alpha_s) \\
&\times\Theta(u-\Delta_B)\frac{1+2v}{u^2M^2}\mathrm{e}^{-\mathrm{EE}}\Big|_{u=(1-v)\alpha_g+\alpha_s} \\
&+2m_d f_K m_K^4\int_0^1 \mathrm{d}v v\int_0^1 \mathrm{d}\alpha_g\int_0^{\alpha_g}\mathrm{d}\beta\int_0^{1-\beta}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\beta,\beta,\alpha)\Theta(u-\Delta_B)}{u^3M^4}\mathrm{e}^{-\mathrm{EE}}\Big|_{1-v\alpha_g} \\
&-2m_d f_K m_K^4\int_0^1 \mathrm{d}v\int_0^1 \mathrm{d}\alpha_g\int_0^{1-\alpha_g}\mathrm{d}\alpha_s\int_0^{\alpha_s}\mathrm{d}\alpha \\
&\times\frac{\Phi(1-\alpha-\alpha_g,\alpha_g,\alpha)\Theta(u-\Delta_B)}{u^3M^4}\mathrm{e}^{-\mathrm{EE}}\Big|_{u=(1-v)\alpha_g+\alpha_s}, \qquad (16)
\end{aligned}
$$

where

$$
\mathrm{DD}=\frac{m_s^2+u(1-u)m_\pi^2-(1-u)q^2}{uM^2},
$$
$$
\mathrm{EE}=\frac{m_d^2+u(1-u)m_K^2-(1-u)q^2}{uM^2},
$$

$$\Delta_A = \frac{m_s^2 - q^2}{s_K^0 - q^2}\,,$$
$$\Delta_B = \frac{m_d^2 - q^2}{s_\pi^0 - q^2}\,. \tag{17}$$

Here s_K^0 and s_π^0 are threshold parameters for the interpolating currents $J_K(x)$ and $J_\pi(x)$, respectively.

3 Numerical results and discussion

The input parameters of the light-cone distribution amplitudes are taken as $\lambda_3 = 1.6 \pm 0.4$, $f_{3K} = (0.45 \pm 0.15) \times 10^{-2}\,\mathrm{GeV}^2$, $\omega_3 = -1.2 \pm 0.7$, $\omega_4 = 0.2 \pm 0.1$, $a_2 = 0.25 \pm 0.15$, $a_1 = 0.06 \pm 0.03$, $\eta_4 = 0.6 \pm 0.2$ for the K meson; $\lambda_3 = 0.0$, $f_{3\pi} = (0.45 \pm 0.15) \times 10^{-2}\,\mathrm{GeV}^2$, $\omega_3 = -1.5 \pm 0.7$, $\omega_4 = 0.2 \pm 0.1$, $a_2 = 0.25 \pm 0.15$, $a_1 = 0.0$, $\eta_4 = 10.0 \pm 3.0$ for the π meson [10–16, 21–23]; and $m_s = (137 \pm 27)\,\mathrm{MeV}$, $m_u = m_d = (5.6 \pm 1.6)\,\mathrm{MeV}$, $f_K = 0.160\,\mathrm{GeV}$, $f_\pi = 0.130\,\mathrm{GeV}$, $m_K = 498\,\mathrm{MeV}$, $m_\pi = 135\,\mathrm{MeV}$. The threshold parameters are chosen to be $s_K^0 = 1.1\,\mathrm{GeV}^2$ and $s_\pi^0 = 0.8\,\mathrm{GeV}^2$, which can reproduce the values of the decay constants $f_K = 160\,\mathrm{MeV}$ and $f_\pi = 130\,\mathrm{MeV}$ in the QCD sum rules.

The Borel parameters in the four sum rules (see (13)–(16)) are taken as $M^2 = (1–2)\,\mathrm{GeV}^2$; in this region, the values of the form factor $f^+_{K\pi}(Q^2)$ from (13) and (14) and $f^+_{K\pi}(Q^2) - f^-_{K\pi}(Q^2)$ from (15) are rather stable, and the values of $f^+_{K\pi}(Q^2) - f^-_{K\pi}(Q^2)$ from (16) are not as stable as the ones from (13)–(15), which are shown, for example, in Figs. 1 and 2, respectively. In this article, we take the special value $M^2 = 1.5\,\mathrm{GeV}^2$ in the numerical calculations, although such a definite Borel parameter cannot take into account some uncertainties, and the predictive power cannot be impaired qualitatively.

The uncertainties of the seven parameters f_{3K} $(f_{3\pi})$, a_2, a_1, λ_3, ω_3, ω_4 and η_4 can only result in small uncertainties for the numerical values. The main uncertainties come from the two parameters m_s and $m_q (= m_u = m_d)$; the variations of those parameters can lead to large changes for the numerical values, which are shown, for example, in Figs. 3 and 4, respectively.

From the two sum rules in (13) and (14), we can see that due to the pseudoscalar currents we choose to interpolate the K and π mesons; the main contributions come from the two-particle twist-3 light-cone distribution amplitudes, not the twist-2 light-cone distribution amplitudes. Those channels can be used to evaluate the non-perturbative parameters in the twist-3 light-cone distribution amplitudes with

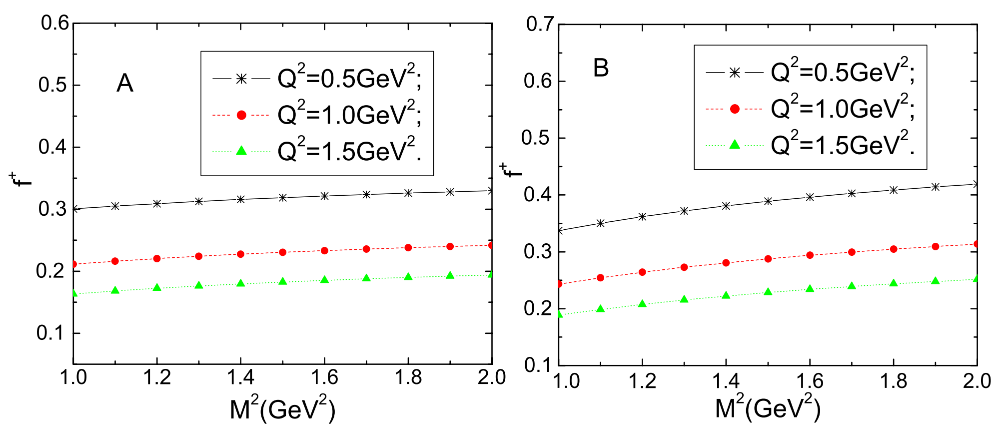

Fig. 1. $f^+_{K\pi}(Q^2)$ as a function of the parameter M^2, **A** from (13) and **B** from (14)

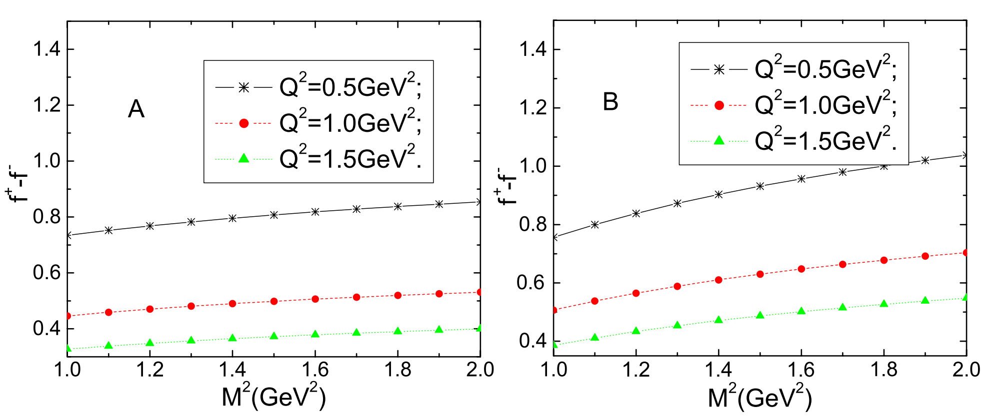

Fig. 2. $f^+_{K\pi}(Q^2) - f^-_{K\pi}(Q^2)$ as a function of the parameter M^2, **A** from (15) and **B** from (16)

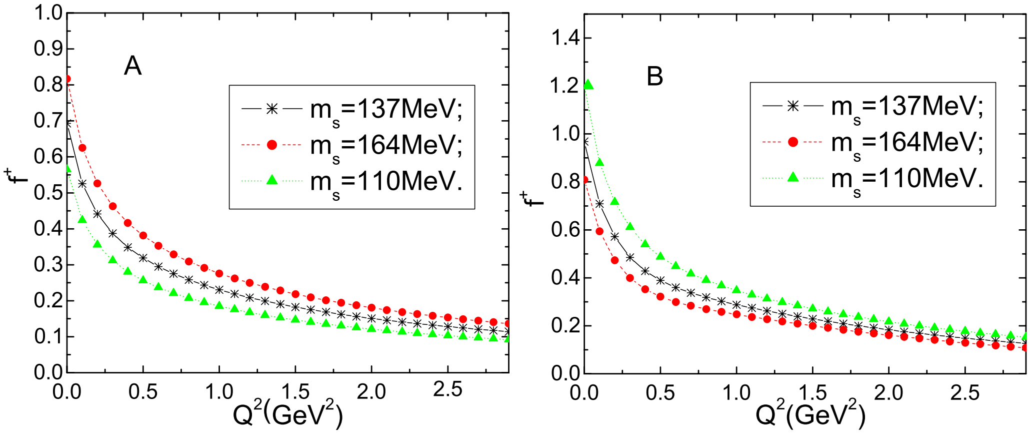

Fig. 3. $f^+_{K\pi}(Q^2)$ with the parameter m_s, **A** from (13) and **B** from (14)

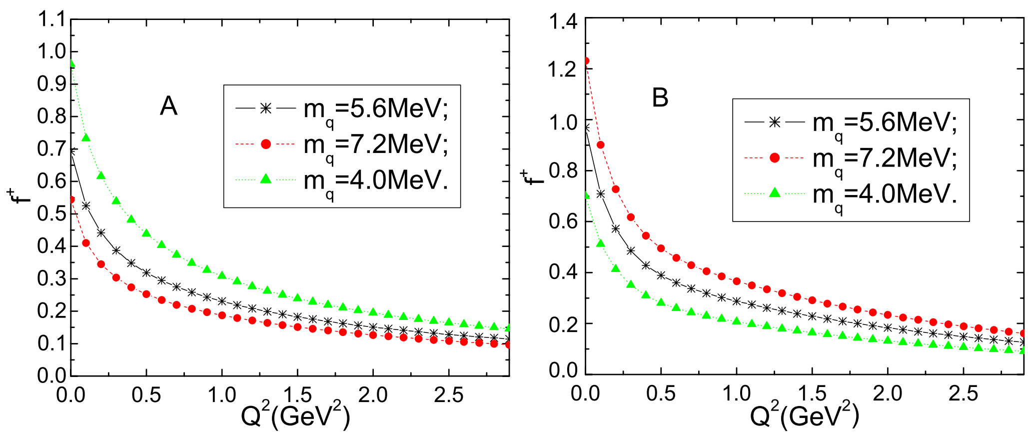

Fig. 4. $f^+_{K\pi}(Q^2)$ with the parameter m_q, **A** from (13) and **B** from (14)

the experimental data. The dominating contributions to the nucleons light-cone distribution amplitudes come from the three valence quarks; additional contributions from the gluons and quark–antiquark pairs are very small [24], and the main contributions to the pseudoscalar mesons light-cone distribution amplitudes come from the two valence quarks – the two cases are analogous. In the light-cone QCD sum rule approach, we can neglect the contributions from the light-cone distribution amplitudes with an additional valence gluon (or quark–antiquark pair) and make relatively rough estimations. For the heavy–light form factors $B \to \pi, K$, we use the pseudoscalar current to interpolate the B meson in the framework of the light-cone QCD sum rule approach. The current mass of the b quark is very large, we can take the chiral limit for the masses of the K and π mesons [25–28], and the contributions from the two-particle twist-3 light-cone distribution amplitudes are very small and can safely be neglected; the analytical expressions are simple. If we use the axial-vector currents to interpolate the K and π mesons, the tensor structures are more complex, and some structures will get dominating contributions from the twist-2 light-cone distribution amplitudes.[1]

We obtain the values of $f^+_{K\pi}(Q^2)$ from the two sum rules in (13) and (14), and then, taking those values as input parameters, we can obtain $f^-_{K\pi}(Q^2)$ from the two sum rules in (15) and (16).

Taking into account all the uncertainties, finally we obtain the numerical values of the form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ that are shown in Fig. 5, at zero momentum transfer:

$$\begin{aligned} f^+_{K\pi}(0) &= 0.69^{+0.30}_{-0.20}\,, \\ f^+_{K\pi}(0) &= 0.97^{+0.35}_{-0.31}\,, \\ \left|f^-_{K\pi}(0)\right| &= 5.15^{+1.59}_{-0.91}\,, \\ \left|f^-_{K\pi}(0)\right| &= 11.47^{+4.98}_{-4.44}\,, \end{aligned} \tag{18}$$

from (13)–(16) respectively. From Fig. 5, we can see that the uncertainties are rather large; we should refine the input parameters m_s and m_q, especially m_q, to improve the predictive ability.

The form factors $f^+_{K\pi}(q^2)$ and $f^0_{K\pi}(q^2)$[2] are measured in $K_{\ell 3}$ decays with the squared momentum transfer to

[1] The results with the axial-vector currents will be presented elsewhere.

[2] We have $f^0_{K\pi}(q^2) = f^+_{K\pi}(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f^-_{K\pi}(q^2)$; the current algebra predicts the value of the scalar form factor $f^0_{K\pi}(\Delta)$

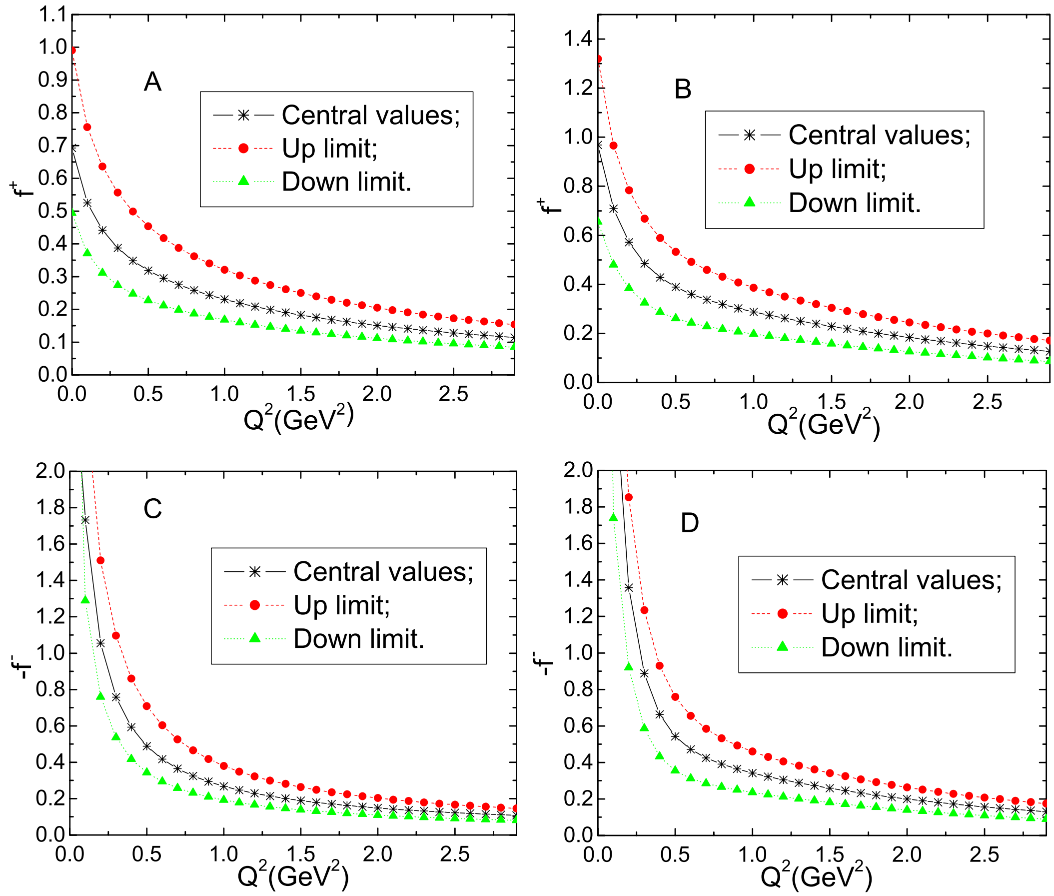

Fig. 5. The values of $f^{+}_{K\pi}(Q^2)$ and $f^{-}_{K\pi}(Q^2)$, **A** from (13), **B** from (14), **C** from (15) and (13), and **D** from (16) and (14)

the leptons q^2, where $q^2 > m_l^2$. The curves (or shapes) of the form factors are always parameterized by the linear model, and by quadratic and pole models to carry out the integrals in the phase space, and the normalization is always chosen to be $f^{+}_{K\pi}(0)$, i.e. $f^{+}_{K\pi}(q^2) = f^{+}_{K\pi}(0)\{1 + \lambda_1 q^2 + \lambda_2 q^4 + \dots\}$, etc. The parameters $\lambda_1, \lambda_2, \dots$ can be fitted by χ^2, etc. [30,31]. From the experimental data, we can obtain the values of the $f^{+}_{K\pi}(0)|V_{us}|$; the basic parameter $f^{+}_{K\pi}(0)$ has to be calculated with some theoretical approaches to extract the CKM matrix element $|V_{us}|$. Comparing with the theoretical calculations from the ChPT [5–9] and lattice QCD [32–36], the central value of $f^{+}_{K\pi}(0)$ ($f^{+}_{K\pi}(0) = 0.97$) from (14) is excellent, while the value $f^{+}_{K\pi}(0) = 0.70$ from (13) is somewhat smaller. The vector form factor $f^{+}_{K\pi}(Q^2)$ has been calculated by ChPT [5–9], lattice QCD [32–36], QCD sum rules [37], the Bethe–Salpeter equation [38], etc. The numerical values $f^{+}_{K\pi}(0) = 0.97^{+0.35}_{-0.31}$ from (14) are more reasonable than the ones from (13).

In Figs. 6 and 7, we plot the form factors $f^{+}_{K\pi}(Q^2)$ and $f^{-}_{K\pi}(Q^2)$ at the momentum range $Q^2 = (0\text{–}12)\,\mathrm{GeV}^2$; from the figures, we can see that the curve (or shape) of $Q^2 f^{+}_{K\pi}(Q^2)$ from (13) is rather flat at $Q^2 > 5\,\mathrm{GeV}^2$, which means that at large momentum transfer, $f^{+}_{K\pi}(Q^2)$ takes the asymptotic behavior $f^{+}_{K\pi}(Q^2) \sim \frac{1}{Q^2}$. It is expected from naive power counting rules [39,40], that the terms proportional to $\frac{1}{Q^{2n}}$ with $n \geq 2$ cancel. The scalar form factor, the axial form factor and the induced pseudoscalar form factor of the nucleons show $\frac{1}{Q^4}$ behavior at large Q^2 [41,42], which is also expected from naive power counting rules [39,40]. The curve (or shape) of $Q^2 f^{+}_{K\pi}(Q^2)$ from (14) at $Q^2 < 5\,\mathrm{GeV}^2$ is analogous to the electromagnetic form factors of the K and π mesons [43].[3] Because of the SU(3) symmetry of the light flavor quarks, we expect that the vector form factor $f^{+}_{K\pi}(Q^2)$ will not be very different from the electromagnetic form factors of the K and π mesons [44–49]; the results from (14) at low Q^2 are more reasonable than the ones from (13). The electromagnetic form factors of the K and π mesons have been calculated with the Bethe–Salpeter equation [47–49], ChPT [52], QCD sum rules [37,44–46], perturbative QCD [50,51,53,54], etc.; our numerical values are compatible with those theoretical calculations. At large mo-

to be $f^{0}_{K\pi}(\Delta) = -f_K/f_\pi$ at the Callan–Treiman point $\Delta = m_K^2 - m_\pi^2$ [29].

[3] One may consult the thesis [43] for more literature on the present state of experimentally determined electromagnetic form factor data of π, K and the proton.

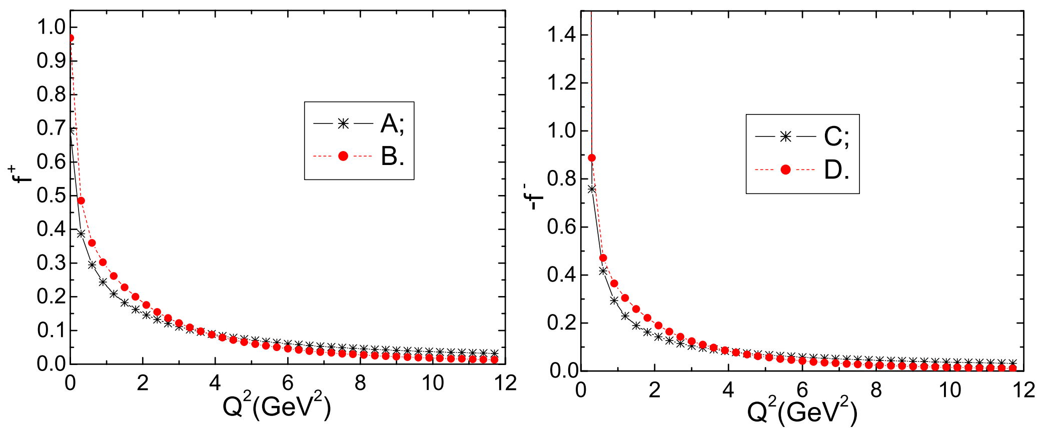

Fig. 6. The central values of $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ at $Q^2 = (0\text{–}12)\,\mathrm{GeV}^2$, **A** from (13), **B** from (14), **C** from (15) and (13), and **D** from (16) and (14)

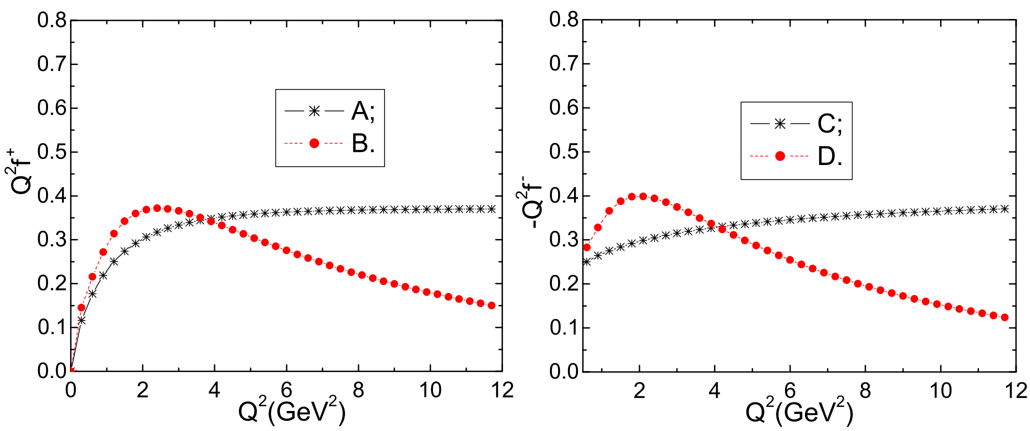

Fig. 7. The central values of $Q^2 f^+_{K\pi}(Q^2)$ and $Q^2 f^-_{K\pi}(Q^2)$ at $Q^2 = (0\text{–}12)\,\mathrm{GeV}^2$, **A** from (13), **B** from (14), **C** from (15) and (13), and **D** from (16) and (14)

mentum transfer with $Q^2 > 5\,\mathrm{GeV}^2$, the terms of $f^+_{K\pi}(Q^2)$ proportional to $\frac{1}{Q^{2n}}$ with $n \geq 2$ from (14) manifest themselves, which results in the curve (or shape) of $Q^2 f^+_{K\pi}(Q^2)$ decreasing with increasing Q^2. The curve (or shape) of the form factor $f^+_{K\pi}(Q^2)$ from (14) decreases more quickly than the one from (13) with increasing Q^2.

The numerical values $|f^-_{K\pi}(0)| = 5.15$ from (15) and $|f^-_{K\pi}(0)| = 11.47$ from (16) are very large compared to the values from the experimental data [30, 31], ChPT [5–9], lattice QCD [32–36] and the Bethe–Salpeter equation [38], and they cannot give any reliable predictions. It is not unexpected, from the sum rules in (15) and (16), that we can see terms like

$$\int_{\Delta_A}^1 \mathrm{d}u \frac{\phi_p(u)}{u} \mathrm{e}^{-\frac{m_s^2 + u(1-u)m_\pi^2 - (1-u)q^2}{uM^2}},$$
$$\int_{\Delta_B}^1 \mathrm{d}u \frac{\phi_p(u)}{u} \mathrm{e}^{-\frac{m_d^2 + u(1-u)m_K^2 - (1-u)q^2}{uM^2}}.$$

$f^-_{K\pi}(Q^2)$ is greatly enhanced in the region of small Q^2 due to the extra $\frac{1}{u}$ compared to the corresponding $f^+_{K\pi}(Q^2)$ in (13) and (14), and in the limit $Q^2 = 0$, $\Delta_A \approx 0.017$ and $\Delta_B \approx 0.00004$, and the dominant contributions come from the end-point of the light-cone distribution amplitudes. Such an infrared behavior can spoil the extrapolation; we should introduce extra phenomenological form factors (for example, the Sudakov factor [53, 54]) to suppress the contribution from the end-point, as we hope to study in our next work. Our numerical values for $f^-_{K\pi}(Q^2)$ have a negative sign compared to $f^+_{K\pi}(Q^2)$, which is consistent with existing theoretical calculations and experimental data. In Figs. 6 and 7, we plot the form factor $f^-_{K\pi}(Q^2)$ at the momentum range $Q^2 = (0\text{–}12)\,\mathrm{GeV}^2$; from the figures, we can see that the form factor $Q^2 f^-_{K\pi}(Q^2)$ from (15), just like $Q^2 f^+_{K\pi}(Q^2)$ from (13), is rather flat at $Q^2 > 5\,\mathrm{GeV}^2$, which means that, at large momentum transfer, $f^-_{K\pi}(Q^2) \sim \frac{1}{Q^2}$. This is also expected from naive power counting rules [39, 40], and the terms proportional to $\frac{1}{Q^{2n}}$ with $n \geq 2$ cancel. At momentum transfer with $Q^2 > 5\,\mathrm{GeV}^2$, the terms of $f^-_{K\pi}(Q^2)$ from (16) proportional to $\frac{1}{Q^{2n}}$ with $n \geq 2$ manifest themselves, which results in the values of $Q^2 f^-_{K\pi}(Q^2)$ (just like $Q^2 f^+_{K\pi}(Q^2)$ from (14)) decreasing with increasing Q^2.

In the light-cone QCD sum rule approach, we carry out the operator product expansion near the light-cone $x^2 \approx 0$, which corresponds to $Q^2 \gg 0$ and $P^2 \gg 0$. The four sum rules in (13)–(16) can be taken as functions that model the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ at large momentum transfer, and we extrapolate $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ to zero momentum transfer or beyond with an analytical continuation.[4] The chosen functions may have good or bad behavior at lower Q^2, which corresponds to the systematic errors, and more experimental data are needed to select the ideal ones.

4 Conclusions

In this article, we calculate the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ within the framework of the light-cone QCD sum rule approach. $f^+_{K\pi}(0)$ is the basic input parameter in extracting the CKM matrix element $|V_{us}|$ from $K_{\ell 3}$ decays. The numerical values of $f^+_{K\pi}(Q^2)$ are compatible with existing theoretical calculations, and the central value of $f^+_{K\pi}(0) = 0.97$ is in excellent agreement with the values from ChPT and lattice QCD. The values of $|f^-_{K\pi}(0)|$ are very large compared to the theoretical calculations and experimental data, and they cannot give any reliable predictions. At large momentum transfer with $Q^2 > 5\,\mathrm{GeV}^2$, the form factors $f^+_{K\pi}(Q^2)$ and $|f^-_{K\pi}(Q^2)|$ can either take up the asymptotic behavior of $\frac{1}{Q^2}$ or decrease more quickly than $\frac{1}{Q^2}$; more experimental data are needed to select the ideal sum rules.

Acknowledgements. This work is supported by National Natural Science Foundation, Grant Number 10405009, and Key Program Foundation of NCEPU.

[4] We can borrow some ideas from the electromagnetic π–photon form factor $f_{\gamma^*\pi^0}(Q^2)$; the value of $f_{\gamma^*\pi^0}(0)$ is fixed by partial conservation of the axial current and the effective anomaly Lagrangian, $f_{\gamma^*\pi^0}(0) = \frac{1}{\pi f_\pi}$, and in the limit of large Q^2, perturbative QCD predicts that $f_{\gamma^*\pi^0}(Q^2) = 4\pi f_\pi/Q^2$. The Brodsky–Lepage interpolation formula [55]

$$f_{\gamma^*\pi^0}(Q^2) = \frac{1}{\pi f_\pi\left[1+Q^2/(4\pi^2 f_\pi^2)\right]} = \frac{1}{\pi f_\pi(1+Q^2/s_0)}$$

can reproduce both the value at $Q^2 = 0$ and the behavior at large Q^2. The energy scale s_0 ($s_0 = 4\pi^2 f_\pi^2 \approx 0.67\,\mathrm{GeV}^2$) is numerically close to the squared mass of the ρ meson, $m_\rho^2 \approx 0.6\,\mathrm{GeV}^2$. The Brodsky–Lepage interpolation formula is similar to the result of the vector meson dominance approach, $f_{\gamma^*\pi^0}(Q^2) = 1/\{\pi f_\pi(1+Q^2/m_\rho^2)\}$. In the latter case, the calculation is performed at the time-like energy scale $q^2 < 1\,\mathrm{GeV}^2$, and the electromagnetic current is saturated by the vector meson ρ, where the mass m_ρ serves as a parameter determining the pion charge radius. With a slight modification of the mass parameter, $m_\rho = \Lambda_\pi = 776\,\mathrm{MeV}$, the experimental data can be well described by the single-pole formula at the interval $Q^2 = (0–10)\,\mathrm{GeV}^2$ [56]. In [57], the four form factors of $\Sigma \to n$ have satisfactory behaviors at large Q^2 as expected by naive power counting rules, and they have finite values at $Q^2 = 0$; the analytical expressions of the four form factors $f_1(Q^2)$, $f_2(Q^2)$, $g_1(Q^2)$ and $g_2(Q^2)$ are taken as Brodsky–Lepage type interpolation formulae, although they are calculated at rather large Q^2, and the extrapolation to lower energy transfer has no solid theoretical foundation. The numerical values of $f_1(0)$, $f_2(0)$, $g_1(0)$ and $g_2(0)$ are compatible with experimental data and theoretical calculations (in magnitude). In this article, the vector form factors $f^+_{K\pi}(Q^2)$ and $f^-_{K\pi}(Q^2)$ can also be taken as Brodsky–Lepage type interpolation formulae; for low momentum transfer the Q^2 behaviors may be good or bad.

Appendix

The light-cone distribution amplitudes of the K meson are

$$\begin{aligned}
&\langle 0|\bar u(0)\gamma_\mu\gamma_5 s(x)|K(p)\rangle \\
&= \mathrm{i} f_K p_\mu \int_0^1 \mathrm{d}u \mathrm{e}^{-\mathrm{i}up\cdot x}\left\{\varphi_K(u) + \frac{m_K^2 x^2}{16}A(u)\right\} \\
&\quad + \mathrm{i} f_K m_K^2 \frac{x_\mu}{2p\cdot x}\int_0^1 \mathrm{d}u \mathrm{e}^{-\mathrm{i}up\cdot x}B(u)\,, \\
&\langle 0|\bar u(0)\mathrm{i}\gamma_5 s(x)|K(p)\rangle \\
&= \frac{f_K m_K^2}{m_s+m_u}\int_0^1 \mathrm{d}u \mathrm{e}^{-\mathrm{i}up\cdot x}\varphi_p(u)\,, \\
&\langle 0|\bar u(0)\sigma_{\mu\nu}\gamma_5 s(x)|K(p)\rangle \\
&= \mathrm{i}(p_\mu x_\nu - p_\nu x_\mu)\frac{f_K m_K^2}{6(m_s+m_u)}\int_0^1 \mathrm{d}u \mathrm{e}^{-\mathrm{i}up\cdot x}\varphi_\sigma(u)\,, \\
&\langle 0|\bar u(0)\sigma_{\alpha\beta}\gamma_5 g_s G_{\mu\nu}(vx)s(x)|K(p)\rangle \\
&= f_{3K}\left\{\left(p_\mu p_\alpha g^\perp_{\nu\beta} - p_\nu p_\alpha g^\perp_{\mu\beta}\right) - \left(p_\mu p_\beta g^\perp_{\nu\alpha} - p_\nu p_\beta g^\perp_{\mu\alpha}\right)\right\} \\
&\quad \times \int \mathcal{D}\alpha_i \varphi_{3K}(\alpha_i)\mathrm{e}^{-\mathrm{i}p\cdot x(\alpha_s+v\alpha_g)}\,, \\
&\langle 0|\bar u(0)\gamma_\mu\gamma_5 g_s G_{\alpha\beta}(vx)s(x)|K(p)\rangle \\
&= p_\mu\frac{p_\alpha x_\beta - p_\beta x_\alpha}{p\cdot x} f_K m_K^2 \int \mathcal{D}\alpha_i A_\parallel(\alpha_i)\mathrm{e}^{-\mathrm{i}p\cdot x(\alpha_s+v\alpha_g)} \\
&\quad + f_K m_K^2(p_\beta g_{\alpha\mu} - p_\alpha g_{\beta\mu})\int \mathcal{D}\alpha_i A_\perp(\alpha_i)\mathrm{e}^{-\mathrm{i}p\cdot x(\alpha_s+v\alpha_g)}\,, \\
&\langle 0|\bar u(0)\gamma_\mu g_s \widetilde G_{\alpha\beta}(vx)s(x)|K(p)\rangle \\
&= p_\mu\frac{p_\alpha x_\beta - p_\beta x_\alpha}{p\cdot x} f_K m_K^2 \int \mathcal{D}\alpha_i V_\parallel(\alpha_i)\mathrm{e}^{-\mathrm{i}p\cdot x(\alpha_s+v\alpha_g)} \\
&\quad + f_K m_K^2(p_\beta g_{\alpha\mu} - p_\alpha g_{\beta\mu})\int \mathcal{D}\alpha_i V_\perp(\alpha_i)\mathrm{e}^{-\mathrm{i}p\cdot x(\alpha_s+v\alpha_g)}\,,
\end{aligned} \tag{A.1}$$

where the operator $\widetilde G_{\alpha\beta}$ is the dual of $G_{\alpha\beta}$, $\widetilde G_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}G^{\mu\nu}$ and $\mathcal{D}\alpha_i$ is defined by $\mathcal{D}\alpha_i = \mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\delta(1-\alpha_1-\alpha_2-\alpha_3)$. The light-cone distribution amplitudes are parameterized by

$$\begin{aligned}
\phi_K(u,\mu) &= 6u(1-u)\left\{1 + a_1 C_1^{\frac{3}{2}}(2u-1) + a_2 C_2^{\frac{3}{2}}(2u-1)\right. \\
&\quad \left. + a_4 C_4^{\frac{3}{2}}(2u-1)\right\}\,,
\end{aligned}$$

$$
\begin{aligned}
\varphi_p(u,\mu) &= 1+\left\{30\eta_3-\frac{5}{2}\rho^2\right\}C_2^{\frac{1}{2}}(2u-1) \\
&\quad +\left\{-3\eta_3\omega_3-\frac{27}{20}\rho^2-\frac{81}{10}\rho^2 a_2\right\}C_4^{\frac{1}{2}}(2u-1)\,, \\
\varphi_\sigma(u,\mu) &= 6u(1-u)\left\{1+\left[5\eta_3-\frac{1}{2}\eta_3\omega_3-\frac{7}{20}\rho^2-\frac{3}{5}\rho^2 a_2\right]\right. \\
&\quad \left.\times C_2^{\frac{3}{2}}(2u-1)\right\}, \\
T(\alpha_i,\mu) &= 360\alpha_u\alpha_s\alpha_g^2\left\{1+\lambda_3(\alpha_u-\alpha_s)+\omega_3\frac{1}{2}(7\alpha_g-3)\right\}, \\
V_\parallel(\alpha_i,\mu) &= 120\alpha_u\alpha_s\alpha_g(v_{00}+v_{10}(3\alpha_g-1))\,, \\
A_\parallel(\alpha_i,\mu) &= 120\alpha_u\alpha_s\alpha_g a_{10}(\alpha_s-\alpha_u)\,, \\
V_\perp(\alpha_i,\mu) &= -30\alpha_g^2\left\{h_{00}(1-\alpha_g)+h_{01}[\alpha_g(1-\alpha_g)-6\alpha_u\alpha_s]\right. \\
&\quad \left.+h_{10}\left[\alpha_g(1-\alpha_g)-\frac{3}{2}(\alpha_u^2+\alpha_s^2)\right]\right\}, \\
A_\perp(\alpha_i,\mu) &= 30\alpha_g^2(\alpha_u-\alpha_s)\left\{h_{00}+h_{01}\alpha_g+\frac{1}{2}h_{10}(5\alpha_g-3)\right\}, \\
A(u,\mu) &= 6u(1-u)\left\{\frac{16}{15}+\frac{24}{35}a_2+20\eta_3+\frac{20}{9}\eta_4\right. \\
&\quad +\left[-\frac{1}{15}+\frac{1}{16}-\frac{7}{27}\eta_3\omega_3-\frac{10}{27}\eta_4\right]C_2^{\frac{3}{2}}(2u-1) \\
&\quad \left.+\left[-\frac{11}{210}a_2-\frac{4}{135}\eta_3\omega_3\right]C_4^{\frac{3}{2}}(2u-1)\right\} \\
&\quad +\left\{-\frac{18}{5}a_2+21\eta_4\omega_4\right\} \\
&\quad \times\{2u^3(10-15u+6u^2)\log u \\
&\quad +2\bar{u}^3(10-15\bar{u}+6\bar{u}^2)\log\bar{u}+u\bar{u}(2+13u\bar{u})\}\,, \\
g_K(u,\mu) &= 1+g_2C_2^{\frac{1}{2}}(2u-1)+g_4C_4^{\frac{1}{2}}(2u-1)\,, \\
B(u,\mu) &= g_K(u,\mu)-\phi_K(u,\mu)\,, \qquad \text{(A.2)}
\end{aligned}
$$

where

$$
\begin{aligned}
h_{00} &= v_{00} = -\frac{\eta_4}{3}\,, \\
a_{10} &= \frac{21}{8}\eta_4\omega_4-\frac{9}{20}a_2\,, \\
v_{10} &= \frac{21}{8}\eta_4\omega_4\,, \\
h_{01} &= \frac{7}{4}\eta_4\omega_4-\frac{3}{20}a_2\,, \\
h_{10} &= \frac{7}{2}\eta_4\omega_4+\frac{3}{20}a_2\,, \\
g_2 &= 1+\frac{18}{7}a_2+60\eta_3+\frac{20}{3}\eta_4\,, \\
g_4 &= -\frac{9}{28}a_2-6\eta_3\omega_3\,; \qquad \text{(A.3)}
\end{aligned}
$$

here $C_2^{\frac{1}{2}}$, $C_4^{\frac{1}{2}}$ and $C_2^{\frac{3}{2}}$ are Gegenbauer polynomials, $\eta_3=\frac{f_{3K}}{f_K}\frac{m_q+m_s}{M_K^2}$ and $\rho^2=\frac{m_s^2}{M_K^2}$ [10–16, 21–23].

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